Period 4, March 4, 2025

$(3 + 5i)(3 - 5i) = 9 - \cancel{15i} + \cancel{15i} - 25i^2 = 9 - 25(-1) = 9 + 25 = 34$

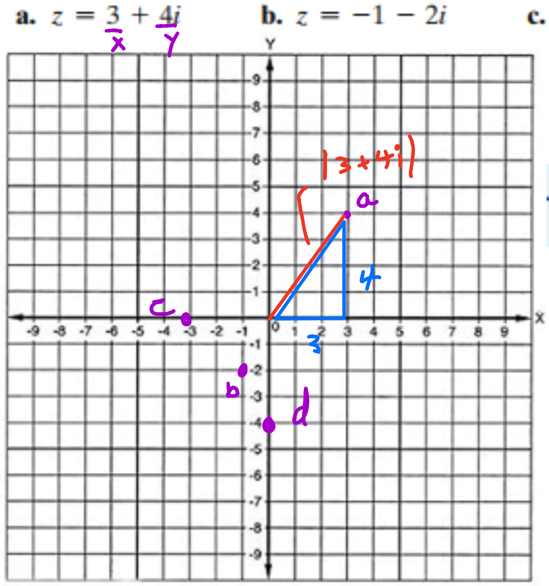
$i^2 = -1$

$$\frac{5 + 2i}{4 - i} = \frac{(5+2i)(4+i)}{(4-i)(4+i)} = \frac{20 + 5i + 8i + 2i^2}{16 + \cancel{4i} - \cancel{4i} - i^2} = \frac{20 + 13i + 2(-1)}{16 - (-1)} = \frac{20 + 13i - 2}{16 + 1} = \frac{18 + 13i}{17}$$

$$= \frac{18}{17} + \frac{13i}{17}$$

Plot each complex number in the complex plane:

a. $z = \underset{x}{3} + \underset{y}{4i}$ **b.** $z = -1 - 2i$ **c.** $z = -3$ **d.** $z = -4i.$



The Absolute Value of a Complex Number

The **absolute value** of the complex number $z = a + bi$ is

a. $z = 3 + 4i$

$\sqrt{3^2 + 4^2} = |3 + 4i|$

$\sqrt{9 + 16}$

$\sqrt{25} = 5$

b. $z = -1 - 2i$

$\sqrt{(-1)^2 + (-2)^2} = |-1 - 2i|$

$\sqrt{1 + 4}$

$\sqrt{5} = |-1 - 2i|$

Polar Form of a Complex Number

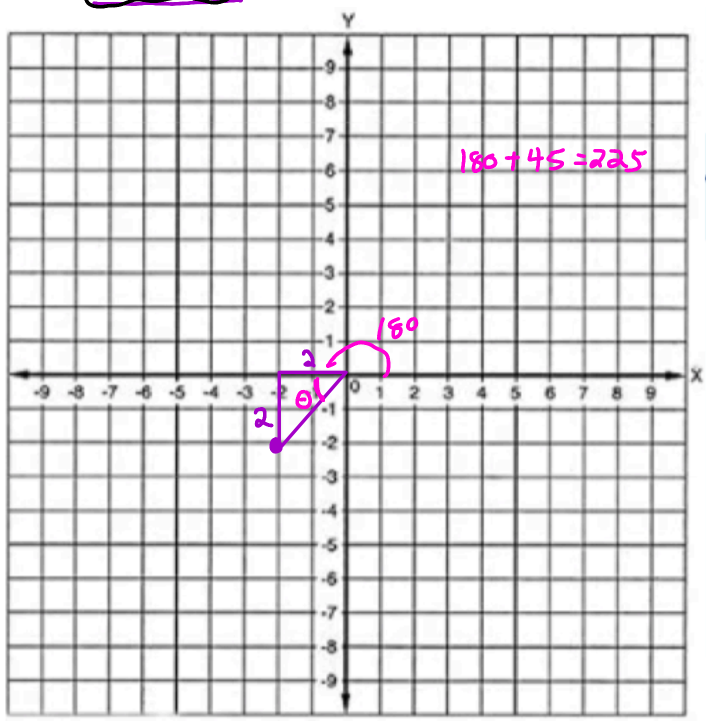
The complex number $z = a + bi$ is written in **polar form** as

$$z = r(\cos\theta + i\sin\theta),$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = \frac{b}{a}$. The value of r is called the **modulus** (plural: moduli) of the complex number z and the angle θ is called the **argument** of the complex number z with $0 \le \theta < 2\pi$.

$(2\sqrt{2}, 225)$ or $\left(2\sqrt{2}, \frac{5\pi}{4}\right)$

Plot $z = -2 - 2i$ in the complex plane. Then write z in polar form.



$|-2-2i| = \sqrt{(-2)^2 + (-2)^2}$

$= \sqrt{4+4}$

$r = \sqrt{8} = 2\sqrt{2}$

$\tan\theta = \frac{2}{2} = 1$

$\tan^{-1} 1 = 45^\circ$ or $\frac{\pi}{4}$

$z = 2\sqrt{2}(\cos 225 + i\sin 225)$

$2\sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right)$

$\frac{2\sqrt{2} \cdot -\sqrt{2}}{2} + i\,\frac{2\sqrt{2} \cdot -\sqrt{2}}{2}$

$\frac{\cancel{2} \cdot -2}{\cancel{2}} + i\,\frac{\cancel{2} \cdot -2}{\cancel{2}} = -2 - 2i$

Write $z = 2(\cos 60^\circ + i\sin 60^\circ)$ in rectangular form.

$= 2\left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$ $\quad (1, \sqrt{3})$

Product of Two Complex Numbers in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers in polar form. Their product, z_1z_2, is

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

To multiply two complex numbers, multiply moduli and add arguments.

$$z_1 \cdot z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$$

$$r_1 \cdot r_2[\cos\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2 + \overset{(-1)}{i^2}\sin\theta_1\sin\theta_2]$$

$$r_1 \cdot r_2[\underbrace{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}_{\cos(\theta_1+\theta_2)} + i(\underbrace{\sin\theta_2\cos\theta_1 + \sin\theta_1\cos\theta_2}_{\sin(\theta_1+\theta_2)})]$$

$$r_1 \cdot r_2(\cos(\theta_1+\theta_2) + i(\sin(\theta_1+\theta_2)))$$

Find the product of the complex numbers. Leave the answer in polar form.

$z_1 = 4(\cos 50° + i\sin 50°)$ $\quad$ $z_2 = 7(\cos 100° + i\sin 100°)$

$$z_1 \cdot z_2 = 4 \cdot 7(\cos(50+100) + i(\sin(50+100)))$$

$$= 28(\cos 150° + i\sin 150°)$$

$$= 28\left(-\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right) = \frac{-28\sqrt{3}}{2} + i \cdot \frac{28}{2} = -14\sqrt{3} + i \cdot 14$$

Quotient of Two Complex Numbers in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers in polar form. Their quotient, $\frac{z_1}{z_2}$, is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

To divide two complex numbers, divide moduli and subtract arguments.

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)} =$$

$$= \frac{r_1(\cos\theta_1\cos\theta_2 - i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2 - \overset{-1}{i^2}(\sin\theta_1\sin\theta_2))}{r_2(\cos^2\theta_2 \cancel{- i\sin\theta_2\cos\theta_2} \cancel{+ i\sin\theta_2\cos\theta_2} - i^2\sin^2\theta_2)}$$

$$= \frac{r_1\left[(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)+i(\sin\theta_1\cos\theta_2-\sin\theta_2\cos\theta_1)\right]}{r_2(\cos^2\theta_2-(-1\sin^2\theta_2))}$$

$$= \frac{r_1\left[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\right]}{r_2(\cos^2\theta_2+\sin^2\theta_2)} = \frac{r_1\left[\cos(\theta_1-\theta_2)+i\sin(\theta_1+\theta_2)\right]}{r_2}$$

Find the quotient $\frac{z_1}{z_2}$ of the complex numbers. Leave the answer in polar form.

$$z_1 = 12\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \qquad z_2 = 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\frac{12\left(\cos\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)\right)}{4} = 3\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

$$3(0+i(1)) = 3i$$

Powers of Complex Numbers in Polar Form

We can use a formula to find powers of complex numbers if the complex numbers are expressed in polar form. This formula can be illustrated by repeatedly multiplying by $r(\cos\theta + i\sin\theta)$.

$z = r(\cos\theta + i\sin\theta)$	Start with z.
$z \cdot z = r(\cos\theta + i\sin\theta)r(\cos\theta + i\sin\theta)$	Multiply z by $z = r(\cos\theta + i\sin\theta)$.
$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$	Multiply moduli: $r \cdot r = r^2$. Add arguments: $\theta + \theta = 2\theta$.

$$z^2 \cdot z = r^2(\cos 2\theta + i\sin 2\theta)r(\cos\theta + i\sin\theta)$$
$$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$$

$$z^3 \cdot z = r^3(\cos 3\theta + i\sin 3\theta)r(\cos\theta + i\sin\theta)$$
$$z^4 = r^4(\cos 4\theta + i\sin 4\theta)$$

$z = r(\cos\theta + i\sin\theta)$

$z^2 = r \cdot r(\cos(\theta+\theta)+i\sin(\theta+\theta))$

$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$

$z^3 = z \cdot z^2 = r(\cos\theta + i\sin\theta) \cdot r^2(\cos 2\theta + i\sin 2\theta)$

$= r \cdot r^2(\cos(\theta+2\theta)+i(\sin(\theta+2\theta)))$

DeMoivre's Theorem

Let $z = r(\cos\theta + i\sin\theta)$ be a complex number in polar form. If n is a positive integer, then z to the nth power, z^n, is

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

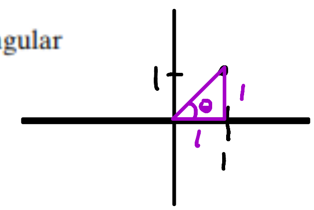
Find $[2(\cos 20° + i\sin 20°)]^6$. Write the answer in rectangular form, $a + bi$.

$$[2(\cos 20° + i\sin 20°)]^6 = 2^6(\cos(20\cdot 6) + i\sin(20\cdot 6)) = 64(\cos 120 + i\sin 120)$$

$$64\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -32 + i32\sqrt{3}$$

Find $(1+i)^8$ using DeMoivre's Theorem. Write the answer in rectangular form, $a + bi$.

$r = \sqrt{1^2+1^2} = \sqrt{2}$



$$(1+i)^8 = [\sqrt{2}(\cos 45° + i\sin 45°)]^8$$

$$= (\sqrt{2})^8[\cos 8(45) + i\sin 8(45)]$$

$$= 16(\cos 360 + i\sin 360)$$

$$16(1 + i(0)) = 16(1+0) = 16$$

$\tan\theta = \frac{1}{1}$

$\tan^{-1} 1 = 45° = \frac{\pi}{4}$

DeMoivre's Theorem for Finding Complex Roots

Let $w = r(\cos\theta + i\sin\theta)$ be a complex number in polar form. If $w \neq 0$, w has n distinct complex nth roots given by the formula

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right] \quad \text{(radians)}$$

$$\text{or} \quad z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right] \text{(degrees)},$$

where $k = 0, 1, 2, \ldots, n-1$.

Find all the complex fourth roots of $16(\cos 120° + i \sin 120°)$. Write roots in polar form, with θ in degrees.

$16\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$$z_0 = \sqrt[4]{16}\left[\cos\left(\frac{120° + 360°\cdot 0}{4}\right) + i \sin\left(\frac{120° + 360°\cdot 0}{4}\right)\right]$$

$$= \sqrt[4]{16}\left(\cos\frac{120°}{4} + i \sin\frac{120°}{4}\right) = 2(\cos 30° + i \sin 30°).$$

$$z_1 = \sqrt[4]{16}\left[\cos\left(\frac{120° + 360°\cdot 1}{4}\right) + i \sin\left(\frac{120° + 360°\cdot 1}{4}\right)\right]$$

$$= \sqrt[4]{16}\left(\cos\frac{480°}{4} + i \sin\frac{480°}{4}\right) = 2(\cos 120° + i \sin 120°).$$

$$z_2 = \sqrt[4]{16}\left[\cos\left(\frac{120° + 360°\cdot 2}{4}\right) + i \sin\left(\frac{120° + 360°\cdot 2}{4}\right)\right]$$

$$= \sqrt[4]{16}\left(\cos\frac{840°}{4} + i \sin\frac{840°}{4}\right) = 2(\cos 210° + i \sin 210°).$$

$$z_3 = \sqrt[4]{16}\left[\cos\left(\frac{120° + 360°\cdot 3}{4}\right) + i \sin\left(\frac{120° + 360°\cdot 3}{4}\right)\right]$$

$$= \sqrt[4]{16}\left(\cos\frac{1200°}{4} + i \sin\frac{1200°}{4}\right) = 2(\cos 300° + i \sin 300°).$$

$[2(\cos 210 + i \sin 210)]^4$

$2^4(\cos(210\cdot 4) + i \sin(210\cdot 4))$

$16(\cos 840 + i \sin 840)$

$840 - 720 = 120$

$16(\cos 120 + i \sin 120) = 16(\cos 840 + i \sin 840)$

$16\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$2(\cos 210 + i \sin 210)$

$\left(2\left(-\frac{\sqrt{3}}{2} + -\frac{1}{2}i\right)\right)^4$

$2^4\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)^2$

$\frac{3}{4} + \frac{i\sqrt{3}}{4} + \frac{i\sqrt{3}}{4} + i^2\cdot\frac{1}{4}$

$\left(\frac{2}{4} + \frac{2i\sqrt{3}}{4}\right) = \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$

$2^4\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$16\left[\frac{1}{4} + \frac{i\sqrt{3}}{4} + \frac{i\sqrt{3}}{4} + i^2\cdot\frac{3}{4}\right]$

$16\left[-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$

Find all the cube roots of 8. Write roots in rectangular form.

$n = 3$

$8 = 8(1 + 0i) = 8(\cos 0 + i \sin 0)$

$\sqrt[3]{8}\left[\cos\left(\frac{0 + 0(360)}{3}\right) + i \sin\left(\frac{0 + 0(360)}{3}\right)\right]$

$2(\cos 0 + i \sin 0) = 2(1 + 0i) = 2$

$\sqrt[3]{8}\left[\cos\left(\frac{0 + 1\cdot 360}{3}\right) + i \sin\left(\frac{0 + 1\cdot 360}{3}\right)\right]$

$2(\cos 120 + i \sin 120) = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -1 + i\sqrt{3}$

$\sqrt[3]{8}\left[\cos\left(\frac{0 + 2\cdot 360}{3}\right) + i \sin\left(\frac{0 + 2\cdot 360}{3}\right)\right]$

$2(\cos 240 + i \sin 240) = 2\left(-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right) = -1 + -i\sqrt{3}$

$\sqrt[3]{8}\left(\cos\left(\frac{0 + 3\cdot 360}{3}\right) + i \sin\left(\frac{0 + 3\cdot 360}{3}\right)\right] = 2(\cancel{\cos 360 + i \sin 360}$